Local Axisymmetric Diffusive Stability of Weakly-Magnetized, Differentially-Rotating, Stratified Fluids

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ABSTRACT

We study the local stability of stratified, differentially-rotating fluids to axisymmetric perturbations in the presence of a weak magnetic field and of finite resistivity, viscosity and heat conductivity. This is a generalization of the Goldreich-Schubert-Fricke (GSF) double-diffusive analysis to the magnetized and resistive, triple-diffusive case. Our fifth-order dispersion relation admits a novel branch which describes a magnetized version of multi-diffusive modes. We derive necessary conditions for axisymmetric stability in the inviscid and perfect-conductor (double-diffusive) limits. In each case, rotation must be constant on cylinders and angular velocity must not decrease with distance from the rotation axis for stability, irrespective of the relative strength of viscous, resistive and heat diffusion. Therefore, in both double-diffusive limits, solid body rotation marginally satisfies our stability criteria. The role of weak magnetic fields is essential to reach these conclusions. The triple-diffusive situation is more complex, and its stability criteria are not easily stated. Numerical analysis of our general dispersion relation confirms our analytic double-diffusive criteria, but also shows that an unstable double-diffusive situation can be significantly stabilized by the addition of a third, ostensibly weaker, diffusion process. We describe a numerical application to the Sun's upper radiative zone and establish that it would be subject to unstable multi-diffusive modes if moderate or strong radial gradients of angular velocity were present.

Subject headings: accretion disks — hydrodynamics — MHD — instabilities — turbulence — Sun: rotation, interior, magnetic fields — stars: rotation

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1. Introduction

Developments in the last decade have made it clear that magnetic fields, even weak magnetic fields, are essential to our understanding of the dynamics of differentially rotating accretion disks and accretion flows in general (Balbus & Hawley 1991; 1998; Balbus 2003; Blaes 2003). Although Keplerian rotation profiles are linearly stable to hydrodynamical axisymmetric perturbations, the introduction of a weak magnetic field, acting as a tether between fluid elements, renders such disks unstable to the magnetorotational instability, or MRI. The ensuing MHD turbulence is now generally viewed as the primary mechanism providing the outward angular momentum transport responsible for accretion in sufficiently ionized, non self-gravitating disks (Hawley, Gammie & Balbus 1995; Armitage 1998; Hawley 2001; Balbus & Hawley 1998).

The destabilizing effect on stellar differential rotation of an embedded magnetic field was studied well before the accretion disk community took notice of its importance. Fricke (1968) pointed out, for example, that time-steady field configurations (isovelocity rotation contours running along field lines) need not be stable. Acheson's thoroughgoing and detailed review (1973) analyzed the effects of toroidal fields, noting that even very weak fields qualitatively change the Goldreich-Schubert-Fricke (Goldreich & Schubert 1967; Fricke 1968) criterion for rotational stability, a topic with which we shall be very much concerned in this paper. The magnetic field problem in its full generality was avoided, however, because of the apparent complications associated with a time-dependent equilibrium field caused by shear. The clear understanding that very general magnetic field configurations that are inconsequential for the equilibrium state can have profound consequences for local WKB perturbations is one of the key conceptual points that emerged in the first accretion disk studies of the MRI (Balbus & Hawley 1991, 1992). In this weak field limit, the time dependence of the unperturbed magnetic field is a nonissue, which opens a broad range of problems to analysis.

The literature on the hydrodynamical stability of stellar differential rotation is vast. The magneto-hydrodynamical stability of differential rotation is a much more specialized topic (e.g., Mestel 1999), and has naturally tended to emphasize the direct dynamical forces associated with the field itself. The Sun and many other stars are expected to possess magnetic fields buried deep inside their radiative zones, as remnants of their complex formation history. Their strength is not well known. It is well known, however, that the time for buried magnetic fields to diffuse out of the solar interior is very long indeed (see, e.g., Parker 1979). Balbus & Hawley (1994) and Balbus (1995; hereafter B95) have studied the linear, adiabatic MRI in a stably-stratified stellar system, and have noted that the strongly-restoring buoyant forces limit the instability to displacements lying only within

spherical shells. The Brunt-Väisälä frequency N in stars is generally orders of magnitude larger than the rotation frequency Ω (a value of 10^3 is typical of the solar radiative zone), whereas N is at best comparable to Ω in a disk. Even more importantly, the most unstable adiabatic MRI modes in a disk are always in the mid-plane (this maximizes the component of the displacement along the angular velocity gradient), and these are insensitive to the vertical stratification profile.

Goldreich & Schubert (1967; hereafter GS) showed that the stabilizing effects of entropy stratification can be compromised by thermal diffusion. The mechanism is analogous to "salt-fingering" in the oceans. In this process, warm salty water overlying cool fresh water, naively a stable configuration, is destabilized by heat transfer. Warm fingers of salty water, penetrating into the cooler waters below, diffuse heat outward more rapidly than they diffuse salt, and thereby loose their buoyancy. In the stellar case, a downwardly displaced fluid element is adiabatically heated and is normally warmer than its ambient surroundings. This results in a restoring buoyant force. But if thermal diffusion causes sufficiently rapid heat leakage, the buoyant force is diminished, and destabilizing angular momentum gradients are then able to operate. GS found that not only must the familiar Rayleigh stability criterion of increasing angular momentum with increasing axial radius be satisfied, the presence of a large thermal conductivity implies that the angular velocity must also be constant along cylindrical axes—a far more stringent criterion.

But great care must therefore be given to apparently small diffusivities when assessing the stability of a stellar rotation profile. The GS result holds when the ratio of the thermal to viscous diffusivities is sufficiently large. Large compared to what? The answer is not unity (Acheson 1978). Rather, it must be large compared to the square of the ratio of the Brunt-Väisälä to rotation frequency, a condition not met for the solar radiative zone. Acheson (1978) also generalized the study of multi-diffusive modes to non-axisymmetric perturbations in a medium with a purely toroidal magnetic field, and found that the angular velocity gradient, not the angular momentum gradient, emerged as the rotational stability discriminant. This result, it turns out, is very general, extending beyond toroidal field geometries (Balbus 1995, 2001).

In this paper, we solve the problem of multi-diffusive stability to axisymmetric perturbations in the presence of rotation, entropy stratification, and a magnetic field of arbitrary geometry. We find that the results are sensitive to the triple combination of resistivity, viscosity, and thermal conductivity, and that all three must be included in the analysis from the very beginning. In what follows, we will refer to such a situation as triple-diffusive, whether it is stable or not.

An outline of the paper is as follows. Section 2 is a derivation of the general dispersion

relation described above. Section 3 is a detailed stability analysis. Section 4 applies the results specifically to the sun's radiative interior. A more general discussion follows in §5, and §6 summarizes our conclusions.

2. Dispersion Relation

The MHD equations including the effects of viscosity, resistivity and heat conduction take the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{1}$$

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho \boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \rho\boldsymbol{\nabla}\Phi + \left(\frac{\boldsymbol{B}}{4\pi}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} + \mu\left(\boldsymbol{\nabla}^2\boldsymbol{v} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\right), \tag{2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \eta\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}), \tag{3}$$

$$\frac{1}{(\gamma-1)}P\frac{d\ln P\rho^{-\gamma}}{dt} = \chi\boldsymbol{\nabla}^2 T. \tag{4}$$

These are, respectively, the continuity equation, the momentum conservation equation, the induction equation and the entropy-form of the energy equation (see, e.g., Balbus & Hawley 1998). Our notation is as follows: $\boldsymbol{v}$ is the flow velocity, ρ is the mass density, P is the pressure, $\boldsymbol{B}$ is the magnetic field, Φ is the gravitational potential, T is the temperature, μ is the dynamic viscosity, η is the resistivity and χ is the heat conductivity (which can represent thermal or radiative conductivity, depending on the problem at hand). In what follows, we write the kinematic viscosity coefficient $\nu = \mu/\rho$. Bulk viscosity effects are neglected. The adiabatic index of the gas, denoted γ, is 5/3 for a monotomic gas with negligible radiation pressure. We have ignored the spatial dependence of the diffusion coefficients μ, η and χ, which is appropriate for a leading order WKB analysis. We have also neglected the resistive and viscous dissipation terms in the entropy equation, which are also higher order terms. The validity of this approximation is examined in Appendix A.

We work in cylindrical coordinates (R, ϕ, Z). We consider axisymmetric Eulerian perturbations (denoted by a prefix δ) with WKB space-time dependence $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, where $\boldsymbol{k} = (k_R, 0, k_Z)$. The basic state magnetic field, allowed to have any geometry, is assumed to be weak enough that it does not affect the basic state configuration, i.e. weak compared to both rotation and pressure gradients. The basic state rotation is given by $\boldsymbol{\Omega} = (0, 0, \Omega(R, Z))$ along the Z–axis. We neglect in our analysis weak circulations such as

those induced by a finite viscosity or any meridional circulation (when considering stellar applications).

Using the Boussinesq approximation, the leading order WKB terms become after linearization

$$k_R\delta v_R + k_Z\delta v_Z = 0, \tag{5}$$

$$\left(-i\omega + \nu k^2\right)\delta v_R \qquad + \frac{ik_R}{\rho}\delta P - 2\Omega\delta v_\phi - \frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial R} + \frac{ik_R}{4\pi\rho} \times \left(B_\phi\,\delta B_\phi + B_Z\,\delta B_Z\right) - \frac{ik_Z}{4\pi\rho}B_Z\,\delta B_R = 0, \tag{6}$$

$$\left(-i\omega + \nu k^2\right)\delta v_\phi + \delta v_R\,\frac{1}{R}\frac{\partial(R^2\Omega)}{\partial R} + \delta v_Z\,R\frac{\partial\Omega}{\partial Z} - i\boldsymbol{k}\cdot\boldsymbol{B}\frac{\delta B_\phi}{4\pi\rho} = 0, \tag{7}$$

$$\left(-i\omega + \nu k^2\right)\delta v_Z \qquad + \frac{ik_Z\,\delta P}{\rho} - \frac{\delta\rho}{\rho^2}\frac{\partial P}{\partial Z} + \frac{ik_Z}{4\pi\rho} \times \left(B_\phi\,\delta B_\phi + B_R\,\delta B_R\right) - \frac{ik_R B_R}{4\pi\rho}\,\delta B_Z = 0, \tag{8}$$

$$\left(-i\omega + \eta k^2\right)\delta B_R - i\boldsymbol{k}\cdot\boldsymbol{B}\delta v_R = 0, \tag{9}$$

$$\left(-i\omega + \eta k^2\right)\delta B_\phi - \delta B_R\,\frac{\partial\Omega}{\partial\ln R} - \delta B_Z\,R\frac{\partial\Omega}{\partial Z} - i\boldsymbol{k}\cdot\boldsymbol{B}\,\delta v_\phi = 0, \tag{10}$$

$$\left(-i\omega + \eta k^2\right)\delta B_Z - i\boldsymbol{k}\cdot\boldsymbol{B}\delta v_Z = 0, \tag{11}$$

$$i\omega\gamma\frac{\delta\rho}{\rho} + (\boldsymbol{\delta v}\cdot\boldsymbol{\nabla})\ln P\rho^{-\gamma} = (\gamma-1)\frac{\chi T k^2}{P}\frac{\delta\rho}{\rho}. \tag{12}$$

The gravitational potential has dropped from the problem because we are interested in wavelengths much shorter than the Jeans wavelength at which self-gravity would become important. The relation $\delta T = -T(\delta\rho/\rho)$ has been used in equation (12) above.

Solving for the eight δ–unknowns in these eight equations, we obtain the following dispersion relation (in compact form)

$$\widetilde{\omega}^4_{b+v}\omega_e\frac{k^2}{k_Z^2} + \widetilde{\omega}^2_{b+v}\omega_b\left[\frac{1}{\gamma\rho}\,(\mathcal{D}P)\,\mathcal{D}\ln P\rho^{-\gamma}\right] + \widetilde{\omega}^2_b\omega_e\left[\frac{1}{R^3}\,\mathcal{D}(R^4\Omega^2)\right] - 4\Omega^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\omega_e = 0, \tag{13}$$

where

$$\boldsymbol{v_A} = \boldsymbol{B}/\sqrt{4\pi\rho}, \qquad k^2 = k_R^2 + k_Z^2, \qquad \widetilde{\omega}^2_{b+v} = \omega_b\omega_v - (\boldsymbol{k}\cdot\boldsymbol{v_A})^2, \qquad \widetilde{\omega}^2_b = \omega_b^2 - (\boldsymbol{k}\cdot\boldsymbol{v_A})^2,$$

$$\omega_b = \omega + i\eta k^2, \qquad \omega_v = \omega + i\nu k^2, \qquad \omega_e = \omega + \frac{\gamma - 1}{\gamma}\frac{iT}{P}\chi k^2, \qquad \mathcal{D} \equiv \left(\frac{k_R}{k_Z}\frac{\partial}{\partial Z} - \frac{\partial}{\partial R}\right)$$

By substituting $\sigma = -i\omega$, this 5th order dispersion relation can be written in developed form

$$a_0\sigma^5 + a_1\sigma^4 + a_2\sigma^3 + a_3\sigma^2 + a_4\sigma + a_5 = 0, \tag{14}$$

where

$$a_0 = k^2/k_Z^2, \tag{15}$$

$$a_1 = \frac{k^2}{k_Z^2}\left[2\nu k^2 + 2\eta k^2 + \xi k^2\right], \tag{16}$$

$$\begin{aligned} a_2 = \frac{k^2}{k_Z^2}&\left[\nu^2 k^4 + \eta^2 k^4 + 4\nu\eta k^4 + 2\nu\xi k^4 + 2\eta\xi k^4 + 2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\right] \\ &- \left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - \left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right], \end{aligned} \tag{17}$$

$$\begin{aligned} a_3 = \frac{k^2}{k_Z^2}&\left[2\eta\nu^2 k^6 + 2\nu\eta^2 k^6 + \nu^2\xi k^6 + \eta^2\xi k^6 + 4\nu\eta\xi k^6 + 2(\nu k^2 + \eta k^2 + \xi k^2)(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\right] \\ &-(2\eta k^2 + \nu k^2)\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - (2\eta k^2 + \xi k^2)\left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right], \end{aligned} \tag{18}$$

$$\begin{aligned} a_4 = \frac{k^2}{k_Z^2}&\left[2\eta\xi\nu^2 k^8 + 2\nu\eta^2\xi k^8 + \eta^2\nu^2 k^8 + 2(\nu\eta k^4 + \nu\xi k^4 + \eta\xi k^4)(\boldsymbol{k}\cdot\boldsymbol{v_A})^2 + (\boldsymbol{k}\cdot\boldsymbol{v_A})^4\right] \\ &-(2\nu\eta k^4 + \eta^2 k^4 + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2)\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] \\ &-(2\eta\xi k^4 + \eta^2 k^4 + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2)\left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right] \\ &-4\Omega^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2, \end{aligned} \tag{19}$$

$$\begin{aligned} a_5 = \frac{k^2}{k_Z^2}&\left[\xi\eta^2\nu^2 k^{10} + 2\xi\nu\eta k^6(\boldsymbol{k}\cdot\boldsymbol{v_A})^2 + \xi k^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^4\right] \\ &-(\nu\eta^2 k^6 + \eta k^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2)\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] \\ &-(\xi\eta^2 k^6 + \xi k^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2)\left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right] \\ &-4\Omega^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\xi k^2, \end{aligned} \tag{20}$$

where, for conciseness, we have introduced the heat diffusivity

$$\xi = \frac{\gamma - 1}{\gamma} \frac{T}{P} \chi.$$

The above dispersion relation reduces to several previously established relations in the appropriate limits. Taking $\nu = \eta = \chi = 0$ (diffusion-free limit), one recovers the result of B95 (his Eq. [2.4]), which is a generalization of the work of Balbus & Hawley (1991; 1994) on the MRI to general rotation laws (i.e. non-constant on cylinders). Taking $v_A = \eta = 0$ (hydrodynamic limit), one recovers exactly the result of Goldreich & Schubert (1967; their Eq. [32]; see also Fricke 1968). Finally, taking $\nu = \eta = 0$ but χ finite, one recovers the result of Balbus (2001; eq. [29]) provided that the isotropic thermal conductivity limit is substituted in his dispersion relation. This is effected by setting $(\boldsymbol{k} \cdot \boldsymbol{b})^2 \rightarrow k^2$ and $\mathcal{D} \ln T \rightarrow 0$ in the Balbus (2001) dispersion relation. (See also Urpin & Brandenburg 1998).

3. Stability Analysis

The complexity of our dispersion relation makes it difficult to derive general necessary *and* sufficient conditions for stability. The single-diffusive case (Balbus 2001) is amenable to a Routh-Hurwitz (RH) analysis, but we have found this method impractical for the multi-diffusive case of interest here. Instead, we derive a series of necessary conditions for stability. As we shall see, they are stringent enough to provide very useful limits on the maximum allowable level of stable differential rotation.

To simplify the analysis further, we first consider the inviscid and perfect-conductor limits separately. This allows us to reduce the problem to two separate double-diffusive situations. As we shall see below, this separation has to be made with some caution. Our stability analysis then proceeds as follows. Given that $a_0 > 0$ in Eq. [14], if any of the five other a_i coefficients is negative, there will be at least one unstable root (i.e. one with a strictly positive real part) to the dispersion relation. A necessary condition for stability is thus that all the a_i be positive. The requirement $a_1 > 0$ is trivially satisfied, so we focus on the criteria corresponding to the positivity of the last four coefficients below. Notice that all the coefficients a_2–a_5 possess first terms in bracket, $\propto k^2/k_Z^2$, that are strictly positive. These terms represent the systematically stabilizing effects of diffusion processes, (ν, η, χ), and magnetic tension, $(\boldsymbol{k} \cdot \boldsymbol{v_A})^2$, on very small scales, where they become dominant. Quite generally, we can focus our attention on (at least somewhat) larger scales for which the coefficients a_2–a_5 depend on terms combining the effects of differential rotation, stratification, diffusion and magnetic tension, and for which positivity is not guaranteed.

In addition, we have found that it is possible to carry out a partial RH analysis. The first non-trivial determinant required to be positive by the RH criterion is $det(2) = a_1 a_2 - a_0 a_3 > 0$. This comes as an additional necessary condition for stability. We have found that this requirement is important to our conclusions on differential rotation.

3.1. Perfect-Conductor Limit ($\eta \rightarrow 0$)

This is the case closest to the analysis of Goldreich & Schubert (1967) and Fricke (1968), who focused on double-diffusive hydrodynamical stability in the presence of viscosity and heat diffusion.

3.1.1. *Requirement $a_2 > 0$ for stability when $\eta \rightarrow 0$*

¿From the structure of the coefficient a_2 (involving pure rotational and stratification terms), we recognize a stability criterion related to diffusion-free, hydrodynamical modes. On large enough scales for the stabilizing effect of the first bracket term in a_2 to be unimportant, the criterion $a_2 > 0$ becomes

$$-\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - \left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right] > 0. \tag{21}$$

As discussed by B95, this inequality translates into the classical Solberg-Høiland criteria (see, e.g., Tassoul 1978)

$$-\frac{1}{\gamma\rho}(\boldsymbol{\nabla}P)\boldsymbol{\cdot}\boldsymbol{\nabla}\ln P\rho^{-\gamma} + \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R} > 0, \tag{22}$$

$$\left(-\frac{\partial P}{\partial Z}\right)\left(\frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R}\frac{\partial \ln P\rho^{-\gamma}}{\partial Z} - \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial Z}\frac{\partial \ln P\rho^{-\gamma}}{\partial R}\right) > 0. \tag{23}$$

For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take a more familiar form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$N^2 + \kappa^2 > 0, \tag{24}$$

$$N^2 \cot\theta \frac{\partial\left(\Omega\sin^2\theta\right)}{\partial\theta} > 0, \tag{25}$$

where the square of the Brunt-Väisälä frequency is defined by

$$N^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial r}\frac{\partial \ln P\rho^{-\gamma}}{\partial r}, \tag{26}$$

and the square of the epicyclic frequency is defined by

$$\kappa^2 = \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R}. \tag{27}$$

Equation (25) shows that to avoid rotational instability in a stratified star, the specific angular momentum must not decrease away from the rotation axis within a spherical shell. While these criteria are necessary and sufficient to guarantee stability in the diffusion–free, hydrodynamical limit, they become only necessary conditions for stability in the broader context studied here.

3.1.2. Requirement $a_3 > 0$ for stability when $\eta \to 0$

¿From the structure of the coefficient a_3, we recognize a stability criterion related to hydrodynamical modes influenced on all scales by viscosity and heat diffusion. This essentially is the constant term of Goldreich & Schubert's third–order dispersion relation. On large enough scales for the stabilizing effect of the first bracket term to be unimportant, the criterion $a_3 > 0$ becomes

$$-\epsilon_\nu\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - \left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right] > 0, \tag{28}$$

where the Prandtl number is defined by

$$\epsilon_\nu = \frac{\gamma\nu}{\gamma-1}\frac{P}{T\chi}. \tag{29}$$

This condition can be rewritten

$$\left(\frac{k_R}{k_Z}\right)^2\epsilon_\nu N_Z^2 + \frac{k_R}{k_Z}\left[\frac{\epsilon_\nu}{\gamma\rho}\left(\frac{\partial P}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial R} + \frac{\partial P}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z}\right) - \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\right]$$
$$+\epsilon_\nu N_R^2 + \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R} > 0, \tag{30}$$

where

$$N_R^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial R}, \qquad N_Z^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z}. \tag{31}$$

As a quadratic in $x = k_R/k_Z$, this inequality is satisfied if and only if (i) the sum of the constant coefficient and the coefficient for x^2 is positive, and (ii) the discriminant of the polynomial in x is negative. By using the vorticity relation for the basic state (e.g. Tassoul 1978)

$$R\frac{\partial\Omega^2}{\partial Z} = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial R}\frac{\partial P}{\partial Z} - \frac{\partial\rho}{\partial Z}\frac{\partial P}{\partial R}\right), \tag{32}$$

to simplify condition (ii), the two criteria can be written

$$\epsilon_\nu(N_R^2 + N_Z^2) + \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R} > 0, \tag{33}$$

$$(1-\epsilon_\nu)^2\left(\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\right)^2 + \frac{4\epsilon_\nu}{\gamma\rho}\frac{\partial P}{\partial Z}\left[\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z} - \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial R}\right] < 0. \tag{34}$$

For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take the following form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$\epsilon_\nu N^2 + \kappa^2 > 0, \tag{35}$$

$$(1-\epsilon_\nu)^2\left(R\frac{\partial\Omega^2}{\partial Z}\right)^2 - 8\epsilon_\nu N^2\Omega\cot\theta\left[\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}\right] < 0. \tag{36}$$

Note that in the limit $\epsilon_\nu \to 0$, these two conditions reduce to the result found by Goldreich & Schubert (1967): necessary conditions for stability are (i) that the specific angular momentum does not decrease with distance from the rotation axis and (ii) that the rotation law be constant on cylinders. Acheson (1978) has pointed out, however, that the limit $\epsilon_\nu \to 0$ must be carefully taken because in some stars, in particular the Sun, the product $\epsilon_\nu N^2$ is not necessarily small compared to the rotational terms. We discuss this limitation further in §4. Interestingly, for $\epsilon_\nu = 1$, the diffusion-free result (§3.1.1) is recovered.

3.1.3. Requirement $a_4 > 0$ for stability when $\eta \to 0$

¿From the structure of the coefficient a_4 (note in particular the $-4\Omega^2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2$ term), we recognize a stability criterion related to diffusion-free, MHD modes. On large enough scales for the stabilizing effect of the first bracket term to be unimportant, the criterion $a_4 > 0$ becomes

$$-\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - \left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2) + 4\Omega^2\right] > 0. \tag{37}$$

Note the additional assumption made to derive this inequality: the weak magnetic field must be strong enough that magnetic tension forces are important on scales larger than those on which dissipation stabilizes all perturbations (e.g. $(\boldsymbol{k} \cdot \boldsymbol{v_A})^2 \gg \eta^2 k^4$). This still leaves a comfortable range of dynamically interesting magnetic field strengths, as is shown for the specific Solar case in Appendix A. As described by B95, the above inequality translates into

$$-\frac{1}{\gamma\rho}(\boldsymbol{\nabla} P)\boldsymbol{\cdot}\boldsymbol{\nabla} \ln P\rho^{-\gamma} + \frac{\partial\Omega^2}{\partial \ln R} > 0, \tag{38}$$

$$\left(-\frac{\partial P}{\partial Z}\right)\left(\frac{\partial\Omega^2}{\partial R}\frac{\partial \ln P\rho^{-\gamma}}{\partial Z} - \frac{\partial\Omega^2}{\partial Z}\frac{\partial \ln P\rho^{-\gamma}}{\partial R}\right) > 0, \tag{39}$$

i.e. forms similar to the classical Solberg-Høiland criteria but with gradients of angular velocity replacing the traditional gradients of specific angular momentum. For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take the following form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$N^2 + \frac{\partial\Omega^2}{\partial \ln R} > 0, \tag{40}$$

$$N^2 \sin\theta\cos\theta\frac{\partial\Omega}{\partial\theta} > 0. \tag{41}$$

Note the important new element introduced by the presence of a magnetic field: to guarantee axisymmetric stability in a stably-stratified star, the angular velocity must not decrease with distance from the rotation axis within a spherical shell. While the two criteria above were necessary and sufficient conditions for stability in the diffusion–free study of B95, they become only necessary conditions for stability in the broader context studied here.

3.1.4. Requirement $a_5 > 0$ for stability when $\eta \to 0$

¿From the structure of the coefficient a_5, we recognize a stability criterion related to MHD modes influenced by viscosity and heat diffusion, not just on small scales. On large enough scales for the stabilizing effect of the first bracket term to be unimportant, the criterion $a_5 > 0$ becomes

$$-\left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2) + 4\Omega^2\right] > 0. \tag{42}$$

Note the remarkable property of this stability condition: it is independent of the stratification term (which drops out of the dispersion relation in the limit $\eta \to 0$). (This property would not be exactly conserved in a triple-diffusive system.) To be satisfied for any combination of k_R and k_Z, it requires the following two conditions

$$\frac{\partial \Omega^2}{\partial \ln R} > 0, \tag{43}$$

$$\left(R \frac{\partial \Omega^2}{\partial Z} \right)^2 < 0, \tag{44}$$

so that marginal stability is possible only for a rotation law that is constant on cylinders ($\partial \Omega / \partial z = 0$).

In a realistic situation, however, even a fully ionized plasma will possess a finite resistivity. One should include resistivity in the analysis, no matter how small it is, because the stratification term can generally be much larger than the rotational term in a stellar context. The situation is intrinsically triple-diffusive because both viscosity and resistivity affect the momentum of displaced fluid elements. This makes the analysis of this branch of the dispersion relation more complicated, and it appears that a simple stability criterion independent of k and v_A does not exist in general.

3.1.5. Requirement $det(2) > 0$ for stability when $\eta \to 0$

Necessary conditions for stability can be made more stringent by the additional requirement $det(2) = a_1 a_2 - a_0 a_3 > 0$. This is one of the five RH determinants that must be positive.

Keeping all the terms in the coefficients a_0 to a_3 when calculating $det(2)$, we note that several are strictly positive and become negligibly small on large enough scales. The condition $det(2) > 0$ can thus be reduced to

$$-(1 + \epsilon_\nu) \left[\frac{1}{\gamma \rho} \mathcal{D} P \, \mathcal{D} \ln P \rho^{-\gamma} \right] - 2 \epsilon_\nu \left[\frac{1}{R^3} \mathcal{D} (R^4 \Omega^2) \right] > 0. \tag{45}$$

Note the different structure of this inequality as compared to Eq. (28). The factor ϵ_ν is now in front of the rotational term (second bracket). In general, for stars, rotational effects are weak compared to the entropy stratification term (first bracket) and $\epsilon_\nu \ll 1$, so that one is tempted to reduce the above expression to the stratification term only. This, however, is

incorrect because terms of order N^2/Ω^2 emerge, and are important. Using the equivalence of Eq. (45) with Eq. (28), if ϵ_ν is replaced by $(1+\epsilon_\nu)/2\epsilon_\nu$ in the latter, the analysis proceeds as in §3.1.2 and we obtain the following two necessary conditions for stability

$$(1+\epsilon_\nu)(N_R^2+N_Z^2)+\frac{2\epsilon_\nu}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R}>0, \tag{46}$$

$$(1-\epsilon_\nu)^2\left(\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\right)^2+\frac{8\epsilon_\nu(1+\epsilon_\nu)}{\gamma\rho}\frac{\partial P}{\partial Z}\left[\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z}-\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial R}\right]<0. \tag{47}$$

For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take the following form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$(1+\epsilon_\nu)N^2+2\epsilon_\nu\kappa^2>0, \tag{48}$$

$$(1-\epsilon_\nu)^2\left(R\frac{\partial\Omega^2}{\partial Z}\right)^2-16\epsilon_\nu(1+\epsilon_\nu)N^2\Omega\cot\theta\left[\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}\right]<0. \tag{49}$$

Note that these conditions differ from those obtained by requiring that $a_3>0$ in § 3.1.2. The first of the two conditions above requires, in the limit $\epsilon_\nu\to0$, that the stratification be stable, this time independently of the rotational stability (measured by κ^2). The second of the two conditions above is more stringent than the corresponding one in § 3.1.2 because it involves a first term that cannot be negative (or, equivalently, is stabilizing). Only the second term can, and its absolute value is a factor of two larger in the above inequalities as compared to those we derived in §3.1.2.

It is worth emphasizing that the combination of requirements $a_2>0$ (§ 3.1.1), $a_3>0$ (§3.1.2) and $det(2)>0$ (above) constitute necessary *and* sufficient stability conditions for the double-diffusive hydrodynamical problem considered by Goldreich & Schubert (1967) and Fricke (1968). One easily shows that an hydrodynamical system satisfying all these conditions also satisfies the RH criterion for the third order dispersion relation of the purely hydrodynamical problem.

3.2. Inviscid Limit ($\nu\to0$)

For the most part, the stability analysis in the limit $\nu\to0$ proceeds in a manner very similar to the limit $\eta\to0$. Unless complications arise, we directly list the necessary

conditions for stability in a convenient form.

3.2.1. Requirement $a_2 > 0$ for stability when $\nu \to 0$

Like in the perfect-conductor limit, the necessary conditions for stability are equivalent to the classical Solberg-Høiland criteria (see §3.1.1).

3.2.2. Requirement $a_3 > 0$ for stability when $\nu \to 0$

On large enough scales, the requirement $a_3 > 0$ becomes, in this case,

$$-2\epsilon_\eta \left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - (2\epsilon_\eta + 1)\left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2)\right] > 0, \tag{50}$$

where the "Acheson number[3]" ϵ_η is

$$\epsilon_\eta = \frac{\gamma\eta}{\gamma - 1}\frac{P}{T\chi}. \tag{51}$$

By similarity with the analysis in §3.1.2, we deduce the following necessary conditions for stability

$$\frac{2\epsilon_\eta}{1+2\epsilon_\eta}(N_R^2 + N_Z^2) + \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R} > 0, \tag{52}$$

$$\begin{aligned}&(1 - \frac{2\epsilon_\eta}{1+2\epsilon_\eta})^2\left(\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\right)^2 \\ &+\frac{4}{\gamma\rho}(\frac{2\epsilon_\eta}{1+2\epsilon_\eta})\frac{\partial P}{\partial Z}\times\left[\frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z} - \frac{1}{R^3}\frac{\partial(R^4\Omega^2)}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial R}\right] < 0.\end{aligned} \tag{53}$$

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[3]We were not able to find an existing terminology for this dimensionless number in the literature. Acheson (1978) appears to have been the first to recognize the importance of this dimensionless quantity for the problem of differential rotation in magnetized and stratified fluids. This number can be expressed, in a rather indirect way, as the ratio of the Prandtl number to the magnetic Prandtl number.

For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take the following form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$\frac{2\epsilon_\eta}{1+2\epsilon_\eta}N^2 + \kappa^2 > 0, \tag{54}$$

$$(1 - \frac{2\epsilon_\eta}{1+2\epsilon_\eta})^2 \left(R\frac{\partial\Omega^2}{\partial Z}\right)^2 - 8(\frac{2\epsilon_\eta}{1+2\epsilon_\eta})N^2\Omega\cot\theta\left[\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}\right] < 0. \tag{55}$$

In the limit $\epsilon_\eta \to 0$, these two conditions are still consistent with the result of Goldreich & Schubert (1967), even though these authors focused on the viscous case, rather than the resistive one. The two necessary conditions for stability are (i) that the specific angular momentum does not decrease with distance from the rotation axis and (ii) that the rotation law be constant on cylinders. The danger in using the limit $\epsilon_\eta \to 0$, as noted by Acheson (1978) for the viscous case, carries over to the resistive case in the sense that the product $\epsilon_\eta N^2/\Omega^2$ is not necessarily small compared to unity (see §4). This time, the diffusion-free criteria (§3.1.1) are not exactly recovered when $\epsilon_\eta = 1$.

3.2.3. Requirement $a_4 > 0$ for stability when $\nu \to 0$

Like in the perfect-conductor limit, the necessary conditions for stability are the "modified Solberg-Høiland" criteria derived by B95 (see §3.1.3).

3.2.4. Requirement $a_5 > 0$ for stability when $\nu \to 0$

On large enough scales, the requirement $a_5 > 0$ becomes

$$-\epsilon_\eta\left[\frac{1}{\gamma\rho}\mathcal{D}P\,\mathcal{D}\ln P\rho^{-\gamma}\right] - \left[\frac{1}{R^3}\mathcal{D}(R^4\Omega^2) + 4\Omega^2\right] > 0. \tag{56}$$

This condition can be rewritten

$$\left(\frac{k_R}{k_Z}\right)^2\epsilon_\eta N_Z^2 + \frac{k_R}{k_Z}\left[\frac{\epsilon_\eta}{\gamma\rho}\left(\frac{\partial P}{\partial Z}\frac{\partial\ln P\rho^{-\gamma}}{\partial R} + \frac{\partial P}{\partial R}\frac{\partial\ln P\rho^{-\gamma}}{\partial Z}\right) - R\frac{\partial\Omega^2}{\partial Z}\right] + \epsilon_\eta N_R^2 + \frac{\partial\Omega^2}{\partial\ln R} > 0. \tag{57}$$

Proceeding as before, the two criteria can be written

$$\epsilon_\eta (N_R^2 + N_Z^2) + \frac{\partial \Omega^2}{\partial \ln R} > 0, \tag{58}$$

$$(1-\epsilon_\eta)^2 \left(R \frac{\partial \Omega^2}{\partial Z} \right)^2 + \frac{4\epsilon_\eta}{\gamma\rho} \frac{\partial P}{\partial Z} \left[\frac{\partial \Omega^2}{\partial \ln R} \frac{\partial \ln P\rho^{-\gamma}}{\partial Z} - R \frac{\partial \Omega^2}{\partial Z} \frac{\partial \ln P\rho^{-\gamma}}{\partial R} \right] < 0. \tag{59}$$

For a star with spherically-symmetric isocontours of density, ρ, and pressure, P, these criteria take the following form when both cylindrical (R, Z) and spherical (r, θ) coordinates are used

$$\epsilon_\eta N^2 + \frac{\partial \Omega^2}{\partial \ln R} > 0, \tag{60}$$

$$(1-\epsilon_\eta)^2 \left(R \frac{\partial \Omega^2}{\partial Z} \right)^2 - 8\epsilon_\eta N^2 \Omega \sin\theta \cos\theta \left[\frac{\partial \Omega}{\partial \theta} \right] < 0. \tag{61}$$

In this case, in the limit $\epsilon_\eta \to 0$, the first of these conditions differs from the result emphasized by Goldreich & Schubert (1967): a necessary condition for stability is that angular velocity (not specific angular momentum) does not decrease with distance from the rotation axis. The second condition, requiring the rotation law to be constant on cylinders for marginal stability, remains the same, however. The diffusion-free result (§3.1.3) is recovered for $\epsilon_\eta = 1$.

3.2.5. *Requirement $det(2) > 0$ for stability when $\nu \to 0$*

In this limit, on large enough scales, the condition $det(2) > 0$ reduces to the simple expression

$$-\left[\frac{1}{\gamma\rho} \mathcal{D}P \, \mathcal{D} \ln P\rho^{-\gamma} \right] > 0, \tag{62}$$

because the various rotational terms cancel out exactly.

This condition can be rewritten

$$\left(\frac{k_R}{k_Z}\right)^2 N_Z^2 + \frac{k_R}{k_Z}\left[\frac{1}{\gamma\rho}\left(\frac{\partial P}{\partial Z}\frac{\partial \ln P\rho^{-\gamma}}{\partial R} + \frac{\partial P}{\partial R}\frac{\partial \ln P\rho^{-\gamma}}{\partial Z}\right)\right] + N_R^2 > 0. \tag{63}$$

Proceeding as before, the two criteria can be written

$$N^2 > 0, \qquad \left(R\frac{\partial \Omega^2}{\partial Z}\right)^2 < 0, \tag{64}$$

so that marginal stability is possible only for a rotation law that is constant on cylinders ($\partial\Omega/\partial Z = 0$). Note that here, the diffusion coefficients η and χ do not explicitly appear. The necessary condition for the rotation law to be constant on cylinders is therefore significantly more stringent than the other $\nu \to 0$ conditions we have derived thus far. Because it relies on the exact cancellation of rotational terms and because it was derived in the inviscid limit, however, this result may not strictly hold in a more realistic triple-diffusive situation, when the fluid possesses a finite, even if small, viscosity. We revisit this issue below when we discuss numerical solutions to our dispersion relation.

3.3. Implications

It is significant that in both the inviscid and perfect-conductor double-diffusive limits, irrespective of the relative strengths of the various diffusion processes involved (within the limitations of our dispersion relation; see Appendix A), rotation must be constant on cylinders for stability. In the perfect-conductor limit, this comes from requiring that $a_5 > 0$. In the inviscid limit, this comes from requiring that the second determinant in the RH analysis be positive ($det(2) > 0$). In both cases, we have also recovered as a necessary condition for stability the result of Balbus & Hawley (1994) and B95: angular velocity must not decrease with distance from the rotation axis within a given spherical shell for stability in the stellar context. Consequently, in both double-diffusive limits, a marginally stable rotation law must at the same time be constant on cylinders and constant within spherical shells. This is achieved only by solid body rotation.[4]

We have cautioned that using one or the other double-diffusive limit can be somewhat

[4]It is important to note here that our focus is on "negative" differential rotation, i.e. differential rotation such that $\partial\Omega/\partial r < 0$ or $\partial\Omega/\partial\theta < 0$ (in terms of the spherical coordinates r and θ). A "positive" differential rotation with, for instance, $\partial\Omega/\partial\theta > 0$ and a rotation constant on cylinders is perfectly stable according to the criteria we derived.

misleading because the addition of a third, even if small, diffusion process should modify the conditions for stability. Because constant rotation on cylinders is related to double-diffusive modes of the type described by Goldreich & Schubert (1967) or to corresponding magnetized modes in our dispersion relation, we intuitively expect the double-diffusive results to be good first approximations as long as focusing on the two largest diffusion processes is justified. It should be so when there is a well defined hierarchy of diffusion processes (e.g. $\nu \ll \eta \ll \xi$). In general, however, it is possible that some finite amount of stable differential rotation persists in a fully triple-diffusive situation. We have not been able to establish necessary conditions for stability in the general, triple-diffusive case and we have addressed this issue by numerically solving the full dispersion relation.

4. Numerical Solutions for the Sun's Radiative Zone

The search for numerical solutions to our dispersion relation described in this section has three goals: (1) to confirm the results of our double-diffusive analysis, (2) to explore the potentially stabilizing role of a third, weaker diffusion process on an otherwise unstable double-diffusive situation, and (3) to apply our results to the Sun's radiative zone.

We adopt a standard model for the current Sun (e.g. Bahcall, Pinsonneault & Basu 2001; Demarque & Guenther 1991). We focus on the part of the radiative zone, from $r \sim 0.3 - 0.7 R_\odot$, in which composition gradients are small. We estimate the values of various microscopic parameters relevant to the stability problem. Following Spitzer (1962), the (ion-dominated) dynamic viscosity for a hydrogen-dominated plasma is

$$\mu = \rho\nu \simeq 2.2 \times 10^{-15} \frac{T^{5/2}}{\ln \Lambda} \text{ g cm}^{-1} \text{ s}^{-1}, \tag{65}$$

where $\ln \Lambda \sim 4$ is an appropriate value of the Coulomb logarithm for the Solar interior. The resistivity for a hydrogen-dominated plasma is

$$\eta \simeq 5.2 \times 10^{11} \frac{\ln \Lambda}{T^{3/2}} \text{ cm}^2 \text{ s}^{-1}. \tag{66}$$

Radiative heat diffusion dominates over thermal heat diffusion in the solar interior, with a radiative conductivity given by (e.g., Schwarzschild 1958)

$$\chi_{rad} = \frac{16 T^3 \sigma}{3 \kappa \rho}, \tag{67}$$

where σ is the Stefan-Boltzmann constant and κ is the radiative opacity. The corresponding radiative diffusivity, which can be directly compared to the kinematic viscosity and resistivity, is given by

$$\xi_{rad} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \chi_{rad}. \tag{68}$$

We list in Table 1 the values of the density, ρ, temperature, T, Rosseland-mean opacity, κ (obtained from a standard opacity table) and all three diffusivities over the region of interest in the solar radiative zone. In addition, the values of the Prandtl and Acheson numbers, ϵ_ν and ϵ_η, which are directly relevant to the stability analysis, are listed. Clearly, resistive diffusion dominates over viscous diffusion in the bulk of the solar radiative zone, by a factor $\sim 20 - 30$. Note that the radiative kinematic viscosity

$$\nu_r = \frac{16}{15} \frac{\sigma T^4}{\kappa \rho^2 c^2}, \tag{69}$$

where c is the speed of light, makes only a small contribution to the total kinematic viscosity in that region (e.g. Goldreich & Schubert 1967). In our specific numerical applications, we focus on the region below the convection zone at $r \lesssim 0.7$, for which helioseismological measurements of a near solid-body rotation are most reliable, with an angular velocity $\Omega = 2.7 \times 10^{-6}$ rad s^{-1} (see, e.g., Charbonneau, Dikpati & Gilman 1999). We adopt the value $N^2 = 1.3 \times 10^{-6}$ Hz^2 for the square of the Brunt-Väisälä frequency (Demarque & Guenther 1991; their Fig. 23), so that the ratio $\Omega^2/N^2 = 5.6 \times 10^{-6}$ in our models.

It is important to note at this point that, in the Sun's radiative zone, the Prandtl number, ϵ_ν, is comparable to Ω^2/N^2 while the Acheson number, ϵ_η, is systematically $\gg \Omega^2/N^2$. Indeed, Acheson (1978) has pointed out that, since the double-diffusive stability criteria of Goldreich & Schubert (1967) are typically of the form $\epsilon_\nu N^2 + \kappa^2 > 0$ (see §3.1.2), substantial differential rotation, for instance a Keplerian-like profile with $\kappa^2 = -\Omega^2$, can remain stable depending on the exact values of ϵ_ν, Ω^2 and N^2. This objection could be a practical limitation of the results of Goldreich & Schubert (1967; see also Fricke 1968), who focused on the idealized limit $\epsilon_\nu \to 0$. Acheson's point is made even stronger by observing that the relevant double-diffusive limit for the Sun is the inviscid one (since $\chi \gg \eta > \nu$) and that $\epsilon_\eta \gg \Omega^2/N^2$ (see Table 1), which appears to be strongly stabilizing by Acheson's arguments. Our extensive stability analysis in the previous section shows, however, that in both the inviscid and perfect-conductor double-diffusive limits, any level of negative differential rotation is destabilized by a combination of diffusion-free and double-diffusive modes, irrespective of the relative strength of viscous, resistive and heat diffusion. This

result effectively invalidates Acheson's objection but it remains to be seen what the effects on stability of adding a third, weaker diffusion process may be.

We have numerically solved the complete dispersion relation (Eq. [14]) with parameters appropriate for the Sun's upper radiative zone, using the Laguerre algorithm described by Press et al. (1992). For definiteness, we have considered only conditions appropriate to the specific radius $r \simeq 0.7R_\odot$: $\nu_\odot = 23.6$ cm^2 s^{-1}, $\eta_\odot = 596$ cm^2 s^{-1} and $\xi_{rad\odot} = 1.2 \times 10^7$ cm^2 s^{-1}. In general, we considered a range of values for the polar angle, θ, in the interval $[0, \pi/2]$ (pole to equator). We have found it useful to rewrite both the rotational and stratification terms appearing in the coefficients a_2–a_5 of the dispersion relation in terms of spherical coordinates (r, θ, ϕ). For a spherical star, the stratification term becomes only function of N^2, θ and the radial and angular wavevectors, k_r and k_θ. The rotational term, on the other hand, explicitly depends on the amount of differential rotation within and between spherical shells, that we express as $\partial \ln \Omega / \partial \theta$ and $\partial \ln \Omega / \partial \ln r$, respectively. As we have mentioned earlier, we are mostly interested in negative differential rotation, i.e. cases where $\partial \ln \Omega / \partial \theta < 0$ and/or $\partial \ln \Omega / \partial \ln r < 0$. We will consider these two cases separately because we expect differential rotation within spherical shells to be destabilized by diffusion-free modes and differential rotation between shells to be destabilized by multi-diffusive modes.

When searching for unstable modes, we vary the wavevectors k_r and k_θ independently in the range

$$\pm\frac{2\pi}{10^{-2}R_\odot} \rightarrow \pm\frac{2\pi}{10^{-14}R_\odot}.$$

The first (large-scale) limit guarantees that we are looking at scales significantly smaller than the pressure scale height ($H_\odot \sim 0.1R_\odot$), while the second (small scale) limit guarantees that we are looking at scales significantly in excess of the mean free path for the conditions of interest. We have also performed focused searches on small regions of the wavevector space corresponding to nearly cylindrical–radial ($|k_R/k_Z| \rightarrow 0$) displacements or nearly vertical ($|k_R/k_Z| \rightarrow \infty$) displacements. Independently of the values of the wavenumbers, we have varied the Alfvèn speeds v_{Ar} and $v_{A\theta}$ in the range 10^{-2} – $10^{-26}R_\odot\Omega$ and we have also explored cases with $v_{Ar} = 0$ and $v_{A\theta} = 0$. This large range of values for the Alfvèn speeds, independently of the values for k_r and k_θ, effectively provides a search both in magnetic field strength and geometry. With this extensive search, we have explored regimes in which $|\boldsymbol{k} \cdot \boldsymbol{v_A}|$ is successively $\gg$, $\ll$ and comparable to each of the three dissipation terms (νk^2, ηk^2 and $\xi_{rad} k^2$).

In the inviscid limit ($\nu = 0$), we were able to find unstable modes down to values of

differential rotation between shells as low as $\partial \ln \Omega / \partial \ln r \lesssim -0.1$ to -0.01, depending on the value of the polar angle, θ. In the perfect-conductor limit ($\eta = 0$), we were able to find unstable modes down to values as low as $\partial \ln \Omega / \partial \ln r \leq -0.01$ for essentially all polar angles. All the unstable modes were of the direct type ($Im(\sigma) = 0$), as opposed to the overstable type ($|Im(\sigma)| > Re(\sigma) > 0$), as expected in the situation of interest, with a strongly stabilizing thermal stratification and a fast rate of heat diffusion. That we found unstable modes down to such very low values of negative differential rotation between shells in both double-diffusive limits is consistent with the conclusions of our stability analysis in §3.

We then turned to the fully triple-diffusive situation and first investigated differential rotation within spherical shells (i.e. $\partial \ln \Omega / \partial \theta < 0$ only). We expect this type of differential rotation to be destabilized by diffusion-free modes, which should be easily identified even in the triple-diffusive case. Indeed, we have found that unstable modes were easily identified down to levels $\partial \ln \Omega / \partial \theta \lesssim -0.1$ to -0.01, depending on the value of the polar angle, θ.

We then searched for unstable modes in the presence of differential rotation between shells (i.e. $\partial \ln \Omega / \partial \ln r < 0$ only). We have been able to identify unstable modes down to levels of differential rotation corresponding to $\partial \ln \Omega / \partial \ln r \simeq -1.2$ to -1.5, depending on the value of the polar angle, θ. It became clear during the extensive search required to identify these triple-diffusive modes that they are located in a much smaller region of the parameter space than the unstable modes we previously identified in the corresponding double-diffusive limit. This is as expected if the addition of a third, weaker diffusion process stabilizes an otherwise unstable double-diffusive situation. We have confirmed this stabilizing effect explicitly by observing that an initially weakly unstable triple-diffusive situation slowly makes a transition to stability as the value of the third, weakest diffusion coefficient is increased.

In that respect, it is worth noting that the triple-diffusive situation in the Sun is such that the weakest diffusion process, viscosity, is “only” a factor 20–30 times smaller than the second weakest one, resistivity, and this may be a source of stabilization with respect to small levels of differential rotation between shells. During our numerical exploration of the triple-diffusive situation in the Sun's upper radiative zone, we have also noticed a rather strong sensitivity of the stability results to small variations (say, $\times 2$) in the values of parameters such as the viscosity, $\nu_\odot$, or the Brunt-Väisälä frequency, N. A complete stability analysis for the current Sun is beyond the scope of the present study. It is encouraging, nonetheless, that we have been able to identify unstable modes in the presence of moderate levels of differential rotation for the current upper radiative zone conditions, as it suggests that these modes may have played an important role in establishing the current

rotation profile in the Sun.

5. The Emergence of Solid Body Rotation

It has long been realized that the timescale for microscopic viscosity to reduce significant levels of differential rotation in the Sun is excessively long ($R_\odot^2/\nu_\odot \sim 0.5\text{-}1 \times 10^{13}$ years; see, e.g., Goldreich & Schubert 1967). On the other hand, differential rotation (with $\partial \ln \Omega / \partial \ln r < 0$) is expected to have been present in the early Sun, because of the likely fast initial rotation and of the magnetic spin-down torque externally exerted via the solar wind (see, e.g., Sofia et al. 1991 for a review). Consequently, the solid body rotation inferred for the Sun's upper radiative zone from seismology (e.g., Kosovichev et al. 1997; Schou et al. 1998; Charbonneau et al. 1999) requires a mechanism capable of reducing differential rotation much more efficiently.

Several mechanisms have been proposed and discussed at length in the literature (see, e.g., Schatzman 1991 for a review). They include the Goldreich-Schubert-Fricke (GSF) instability, the "secular hydrodynamical shear instability" (Zahn 1974) and angular momentum transport by internal gravity waves (Press 1981; Kumar & Quataert 1997; Talon & Zahn 1998; Talon, Kumar & Zahn 2002). None of these mechanisms may be able to provide a satisfactory solution, however. The GSF instability alone is not expected to bring a system to a state of solid body rotation (but only of rotation constant on cylinders; e.g., Goldreich & Schubert 1967). The existence of a secular shear instability has not been rigorously proven, but only inferred from heuristic arguments (Zahn 1974; Schatzman 1991), and it would operate in the current Sun's radiative zone only in the presence of strong shear (Zahn 1993). Finally, while internal gravity waves may transport angular momentum efficiently, they are not generally expected to be efficient at mixing elements. The large amount of Li depletion at the surface of the Sun and other stars (see, e.g., Chaboyer, Demarque & Pinsonneault 1995a,b) is best interpreted as resulting from turbulent mixing in stellar radiative zones, thus favoring the action of instabilities rather than waves to explain both mixing and rotational evolution.

According to our analysis, it is the combination of weak magnetic fields and multi-diffusive modes that may render even a small level of (negative) differential rotation unstable. Independently of this issue of stability, there are two important aspects of the problem that a linear analysis cannot address. First, it is *a priori* unclear whether the turbulence resulting from the non-linear development of the various modes described by our dispersion relation will drive the system towards a state of marginal stability (i.e. near solid body rotation). Intuitively, because the main force balance in a star does not

involve rotation (contrary to an accretion disk), we would expect the turbulence to be relatively "free" to bring a star close to a state of marginal stability, but this remains to be proven. Second, it is also unclear with what efficiency (i.e. on what timescale) that turbulence would be able to transport angular momentum and affect an unstable rotation profile. Addressing these two non-linear aspects of the problem reliably would require fully turbulent numerical simulations. In that respect, it is interesting to note that a preliminary two-dimensional investigation by Korycansky (1991) of hydrodynamical double-diffusive modes in a specific "equatorial" geometry does indicate that angular momentum transport drives the system towards marginal stability. An additional motivation for carrying out such detailed numerical simulations of turbulence driven by multi-diffusive, magnetized and unmagnetized modes would be to estimate the efficiency of turbulent transport of elements.

One important physical element that has been neglected in our analysis is the stabilizing effect of composition gradients. Goldreich & Schubert (1967) have described, in the single-diffusive hydrodynamical limit, how even moderate gradients of chemical composition can stabilize significant levels of radial differential rotation. As these authors have noted, this effect would be important at radii $r \lesssim 0.3 R_\odot$ in the Sun's core, a region that we have excluded from our analysis. While a complete derivation of our dispersion relation including the effects of composition gradients is beyond the scope of the present study, by analogy with the results of Goldreich & Schubert (1967), one may expect significantly stronger levels of radial differential rotation to be maintained in the solar core, relative to the rest of the radiative zone. On the other hand, it is possible that the early differential rotation in the region currently encompassing the solar core was reduced before significant hydrogen burning took place, if multi-diffusive modes were rather efficient early on at redistributing angular momentum. This illustrates how the problem of differential rotation is intimately linked to that of stellar evolution.

6. Conclusion

We have studied the local axisymmetric triple-diffusive stability of stratified, weakly-magnetized, differentially-rotating fluids. We have established that, in an inviscid or a perfectly-conducting fluid, differential rotation is destabilized by a combination of diffusion-free and double-diffusive modes, unless rotation is constant on cylinders and angular velocity does not decrease away from the rotation axis. We have stressed the important role of weak magnetic fields in establishing these results. We have found that, in a more realistic triple-diffusive situation, the weakest diffusion process can sometimes play a stabilizing role. While our analysis is rather general, we have discussed a specific numerical

application to the Sun's upper radiative zone, which is seismologically known to be rotating near solid body rotation. We have found that moderate to strong levels of differential rotation, if present, would indeed be destabilized, thus suggesting that magnetized and multi-diffusive modes may have played an important role in establishing the current solar internal rotation.

Acknowledgments

KM gratefully acknowledges support from the Celerity Foundation. Support for this research was provided by NASA Grants NAG5–13288 and NAG5-9266.

Appendix A: Limitations

A1: Validity of the dispersion relation

We neglected the resistive and viscous dissipation terms in the entropy equation (Eq. [4]; see, e.g., Tassoul 1978 or Balbus & Hawley 1998 for complete formulations) and its subsequent linearized form (Eq. [12]). We explore the range of validity of this approximation here.

Let us first compare the magnitudes of the three terms we kept in equation (12) for entropy perturbations. The ratio of the first term on the LHS to the perturbed heat diffusion term is of order

$$R_1 = \frac{\omega \delta T/T}{\chi k^2 \delta T/P} \sim \frac{\omega}{\chi k^2 T/P} \sim \frac{\omega}{(\xi/\nu)\lambda c_s k^2} \sim \frac{1}{(\xi/\nu)(kH)(k\lambda)}, \tag{70}$$

where we have used the definition of the heat diffusivity, ξ, and we have equated the kinematic viscosity coefficient, ν, to the product of the mean free path, λ, and sound speed, c_s. Note that the perturbation frequency, ω, is typically (though not exclusively) $\sim N$ (the Brunt-Väisälä frequency) in a pressure-supported system and $\sim \Omega$ (the rotation frequency) in a rotation-supported system. In both cases, $\omega \sim c_s/H$, where H is the pressure scale-height of the system. Irrespective of the ratio of diffusivities (ξ/ν), our local analysis making use of the MHD equations is valid only for scales much smaller than the system's scale-height ($kH \gg 1$) and much larger than the mean free path ($k\lambda \ll 1$). In general, the ratio R_1 can therefore be $\gg 1$ or $\ll 1$ and one must keep the heat diffusion term in the perturbed entropy equation to leading order.

The ratio of the same first term on the LHS of equation (12) to the (neglected) perturbed viscous heating term is of order

$$R_2 = \frac{\omega \delta T/T}{\delta[\mu \, |d\Omega/d\ln r|^2]/P} \sim \frac{\omega \delta v/c_s}{\nu \Omega k \delta v/c_s^2} \sim \frac{\omega}{k\lambda\Omega}, \tag{71}$$

where we have used the relations $\delta v/c_s \sim \delta\rho/\rho \sim \delta T/T$ and $|d\Omega/d\ln r| \sim \Omega$. Since ω/Ω is typically ~ 1 (rotation-supported system) or $\gg 1$ (pressure-supported system), and $k\lambda \ll 1$, R_2 is $\gg 1$ and neglecting the perturbed viscous heating term is justified to leading order.

Similarly, the ratio of the first term on the LHS of equation (12) to the (neglected) perturbed resistive heating term is of order

$$R_3 = \frac{\omega \delta T/T}{\delta[(\eta/4\pi) \, |\nabla \times B|^2]/P} \sim \frac{\omega \delta T/T}{(\eta/4\pi)(v_A/c_s)^2 k \delta c_s/(c_s H)} \sim \frac{1}{(\eta/8\pi\nu)(v_A/c_s)^2 (k\lambda)}, \tag{72}$$

where we have made the additional assumption that the ratio v_A/c_s is locally constant in the basic state configuration. This amounts to requiring that the basic state magnetic field does not possess strong gradients on scales smaller than the scale height, H, and is a reasonable approximation unless one is interested in rather singular basic state magnetic field configurations. Irrespective of the ratio of diffusivities, η/ν, our weak field assumption implies that $v_A/c_s \ll 1$, so that R_3 is also $\gg 1$ in general and neglecting the perturbed resistive heating term is justified to leading order.

Physically, this hierarchy of heating terms in the perturbed entropy equation can be understood by noting that heat diffusion is naturally the most efficient form of heat transport, unless the viscosity and/or resistivity coefficients are very much larger than the heat diffusivity coefficient (i.e. ν or $\eta \gg \xi$). That our assumptions for the leading-order perturbed equations remain valid even if ν or η is larger than ξ, but not so much as to become more efficient at transporting heat than heat diffusion itself, indicates that our dispersion relation is able to describe overstable modes in a strongly stratified medium where such a hierarchy of diffusivities occurs.

A2: How weak can weak magnetic fields be?

In the coefficients a_4 and a_5 of our dispersion relation, pre-factors multiplying the rotational and stratification terms involve sums of dissipation and magnetic tension terms. Our double-diffusive stability analysis in §3 assumes that the magnetic field strength, while

weak ($v_A \ll c_s$ and $v_A \ll (HR)^{1/2}\Omega$), is strong enough for the magnetic tension term to dominate over the dissipation terms in all these pre-factors. We determine the range of validity of this assumption here, using numerical values appropriate for the Sun.

Let ξ denote the largest diffusivity coefficient (in units of cm^2 s^{-1}) in the medium of interest. For the assumption made in our double-diffusive stability analysis to be incorrect, we need $k^2 v_A^2 \ll \xi^2 k^4$ for all possible values of k relevant to the local analysis. Since the minimum acceptable value of k is $\sim 2\pi/H_\odot$ (where $H_\odot$ is the scale height), our assumption breaks down for all relevant scales if $v_A^2 \ll 4\pi^2\xi^2/H_\odot^2$. Using typical values $\rho \sim 1$ g cm^{-3}, $\xi \sim 10^7$ cm^2 s^{-1} and $H_\odot/R_\odot \sim 0.1$ for the Sun's upper radiative zone, this translates into a limit on the field strength $B \ll 8 \times 10^{-5}$ G, a very small value indeed. Since the above-mentioned pre-factors in a_4 and a_5 actually involve products of several diffusivity coefficients (not just the largest one), the limit on the field strength will be smaller than we estimated by several extra orders of magnitude for the conditions in the Sun's upper radiative zone. On the other hand, our assumption of weak magnetic field leads to $v_A^2 = B^2/4\pi\rho \ll (H_\odot R_\odot)^{1/2}\Omega \ll c_s^2$ or $B \ll 10^3$ G. This still leaves a comfortable range of field strengths for which the assumptions made in our double-diffusive stability analysis are valid for the solar interior.

REFERENCES

Acheson, D. J. 1978, Phil. Trans. Roy. Soc. London, 289A, 459

Armitage, P. J. 1998, ApJ, 501, L189

Bahcall, J. N., Pinsonneault, M. H. & Basu, S. 2001, ApJ, 555, 990

Balbus, S. A. & Hawley, J. F. 1991, ApJ, 376, 214

Balbus, S. A. & Hawley, J. F. 1994, MNRAS, 266, 769

Balbus, S. A., & Hawley, J. F. 1998, Rev. Mod. Phys., 70, 1

Balbus, S. A. 1995, ApJ, 453, 380 (B95)

Balbus, S. A. 2001, ApJ, 562, 909

Balbus, S. A. 2003, ARA&A, in press (preprint)

Blaes, O. 2003, in "Accretion Disks, Jets, and High Energy Phenomena in Astrophysics", Proceedings of Session LXXVIII of Les Houches Summer School, August 2002, eds. F. Menard, G. Pelletier, G. Henri, V. Beskin, and J. Dalibard (EDP Science: Paris and Springer: Berlin), astro-ph/0211368

Chaboyer, B., Demarque, P. & Pinsonneault, M. H. 1995a, ApJ, 441, 865

Chaboyer, B., Demarque, P. & Pinsonneault, M. H. 1995b, ApJ, 441, 876

Charbonneau, P., Dikpati, M. & Gilman, P. A. 1999, ApJ, 526, 523

Demarque, P. & Guenther, D. B. 1991, in Solar Interior and Atmosphere, eds. A.N. Cox, W.C. Livingston and M.S. Matthews (Tucson: University of Arizona Press), p. 1186

Fricke, K. 1968, Z. Ap., 68, 137

Goldreich, P. & Schubert, G. 1967, ApJ, 150, 571

Hawley, J. F. 2001, ApJ, 554, 534

Hawley, J. F., Gammie, C. F. & Balbus, S. A. 1995, ApJ, 440, 742

Korycansky, D. G. 1991, ApJ, 381, 515

Kosovichev, A. G. et al. 1997, Sol Phys., 170, 43

Kumar, P. & Quataert, E. J. 1997, ApJ, 479, L51

Parker, E. N. 1979, Cosmical Magnetic Fields (New York: Oxford University Press)

Press, W. H., Teukolsky, S. A., Vetterling, W.T. & Flannery, B. P. 1992, Numerical Recipes in Fortran. The art of scientific computing (Cambridge: Cambridge University Press)

Schou, J. et al. 1998, ApJ, 505, 390

Schwarzschild, M. 1958, Structure and Evolution of the Stars (New York: Dover)

Sofia, S., Kawaler, S., Larson, R. & Pinsonneault, M. 1991, in Solar Interior and Atmosphere, eds. A.N. Cox, W.C. Livingston and M.S. Matthews (Tucson: University of Arizona Press), p. 140

Schatzman, E. 1991, in Solar Interior and Atmosphere, eds. A.N. Cox, W.C. Livingston and M.S. Matthews (Tucson: University of Arizona Press), p. 192

Spiegel, E. A. & Zahn, J. P. 1992, A&A, 265, 115

Spitzer, L. 1962, Physics of Fully Ionized Gases (New York: Wiley Interscience)

Talon, S. & Zahn, J.-P. 1998, A&A, 329, 315

Talon, S., Kumar, P. & Zahn, J.-P. 2002, ApJ, 574, L175

Tassoul, J.-L. 1978, Theory of Rotating Stars (Princeton: Princeton University Press)

Urpin, V. & Brandenburg, A. 1998, MNRAS, 294, 399

Zahn, J.-P. 1974, in Stellar Instability and Evolution, Proc. IAU Symp. 129, eds. P. Ledoux, A. Noels and A.W. Rodgers (Dordrecht: D. Reidel), p. 185

Zahn, J.-P. 1993, in Les Houches School (Session XLVII): Astrophysical Fluid Dynamics, eds. J.-P. Zahn & J. Zinn-Justin (Elsevier), p. 561

Table 1: Diffusive Conditions in the Solar Radiative Zone

Radius ($R_\odot$)	ρ (g cm^{-3})	T (10^6 K)	κ (cm^2 g^{-1})	ν (cm^2 s^{-1})	ν_r (cm^2 s^{-1})	η (cm^2 s^{-1})	ξ_{rad} (cm^2 s^{-1})	ϵ_ν	ϵ_η
$r \simeq 0.7$	0.2	2.3	18	21	2.6	596	1.2×10^7	2×10^{-6}	5×10^{-5}
$r \simeq 0.3$	8.5	6.2	3	6	0.5	135	8.1×10^5	8×10^{-6}	1.7×10^{-4}